§1.6 Calculating limits Using the Limit Laws.

Key points: ☆① (Linear) $\boxed{\text{Limit Laws}}$ by general combination

☆☆② Limits by canceling zeros: Factoring technique.

③ Squeeze Theorem.

① Limit Laws:

- Sum / Difference: $\lim_{x\to a}[f(x)\pm g(x)] = [\lim_{x\to a} f(x)] \pm [\lim_{x\to a} g(x)]$
- Constant multiple: $\lim_{x\to a}[c\cdot f(x)] = c\cdot[\lim_{x\to a} f(x)]$, $\lim_{x\to a} c = c$.
- Product: $\lim_{x\to a}[f(x)\cdot g(x)] = [\lim_{x\to a} f(x)]\cdot[\lim_{x\to a} g(x)]$
- Quotient: $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$.
- Power/Root: $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$, $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$.
- Constant / Polynomials: $\lim_{x\to a} c = c$, $\lim_{x\to a} x = a$, $\lim_{x\to a} x^2 = a^2$, $\lim_{x\to a} x^3 = a^3$, ...

  $\lim_{x\to a}\sqrt{x} = \sqrt{a}$ $(a>0)$, $\lim_{x\to a}\frac{1}{x} = \frac{1}{a}$ $(a\neq 0)$, $\lim_{x\to a}\frac{1}{x^2} = \frac{1}{a^2}$, ...

eg1. Suppose $\lim_{x\to 2} f(x) = -1$, $\lim_{x\to 2} g(x) = 3$. Compute the following limits.

- $\lim_{x\to 2}\left[5 - \frac{x^3}{f(x)+g(x)} + 2\cdot[f(x)]^2\cdot\sqrt{g(x)}\right]$

$$= \lim_{x\to 2}[5] - \lim_{x\to 2}\left[\frac{x^3}{f(x)+g(x)}\right] + \lim_{x\to 2}\left[2\cdot(f(x))^2\cdot\sqrt{g(x)}\right]$$

$$= 5 - \frac{\lim_{x\to 2} x^3}{[\lim_{x\to 2} f(x)] + [\lim_{x\to 2} g(x)]} + 2\cdot[\lim_{x\to 2} f(x)]^2\cdot\sqrt{\lim_{x\to 2} g(x)}$$

$$= 5 - \frac{2^3}{-1+3} + 2\cdot(-1)^2\cdot\sqrt{3}$$

$$= 5 - 4 + 2\cdot\sqrt{3} = \boxed{1+2\sqrt{3}}$$ ※

Rmk: All the laws could be applied to one-sided limits.

e.g.2 (Direct plug in)

• $\lim_{x\to 0} \frac{x+1}{3x^2-5x+7} = \frac{0+1}{0+7} = \frac{1}{7}$; • $\lim_{h\to 1} \frac{2h-h^2}{h+1} = \frac{2-1}{1+1} = \frac{1}{2}$; • $\lim_{u\to -3} \sqrt{9-u^2} = \sqrt{9-(-3)^2} = \sqrt{0} = 0$

☆☆② In the quotient form $\lim_{x\to a} \frac{f(x)}{g(x)}$. If both $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then we must cancel out the "zero terms" in $f(x)$ and $g(x)$ by the following factoring technique.

e.g.3. Compute $\lim_{x\to 2} \frac{x^2+x-6}{x^2-2x}$. Remark: If we plug in $x=2$, we get $\frac{2^2+2-6}{2^2-2\cdot 2} = \frac{0}{0}$, which is meaningless.

solution: $\underset{\text{Factorize}}{=} \lim_{x\to 2} \frac{\cancel{(x-2)}\cdot(x+3)}{x\cdot\cancel{(x-2)}} \underset{\text{cancel out}}{=} \lim_{x\to 2} \frac{x+3}{x} \underset{\text{Plug in}}{=} \frac{2+3}{2} = \boxed{\frac{5}{2}}$

e.g.4. (S16) If $f(x) = \frac{1}{x+3}$, then $\lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{-1}{4(x+3)} \underset{\text{Plug in}}{=} \frac{-1}{4\cdot 4} = \boxed{-\frac{1}{16}}$

Solution: $\frac{f(x)-f(1)}{x-1} = \frac{\frac{1}{x+3}-\frac{1}{4}}{x-1} = \frac{\frac{4-x-3}{4(x+3)}}{x-1} = \frac{1-x}{4(x+3)(x-1)} = \frac{-1}{4\cdot(x+3)}$ Hint: $\frac{\frac{a}{b}}{c} = \frac{a}{b\cdot c}$

e.g.5. (sec 1.5) Compute $\lim_{x\to -3^+} \frac{x-2}{x^2\cdot(x+3)} = -\infty$. Hint: $x-2 = (-3)-2 = -5$.
$x^2 = (-3)^2 = +9$.
$x+3 = (-3)+3 = 0$

$\left(\frac{-5}{9\cdot(\text{small positive})} = -\infty \right)$ while $x+3$ small but positive since $x \to -3^+$ (approaches -3 from right)

☆ e.g.6. (F16) For what value of c does $\lim_{x\to 2} \frac{cx^2+4}{x-2}$, exist and is finite?

Solution: Notice if we plug in $x=2$, we have $\frac{4c+4}{2-2}$, where the denominator is 0.

So we need the numerator also be zero, i.e., $4c+4=0 \Rightarrow \boxed{c=-1}$

Actually, if $c=-1$, $\lim_{x\to 2} \frac{(-1)\cdot x^2+4}{x-2} = \lim_{x\to 2} \frac{(2+x)\cdot(2-x)}{x-2}$ Hint: $a^2-b^2=(a+b)(a-b)$

$= \lim_{x\to 2} \frac{-(2+x)\cdot\cancel{1}}{\cancel{1}}$ $2^2-x^2 = 4-x^2 = (2+x)(2-x)$

Therefore, $\boxed{c=-1}$

$= -4$.

③ Squeeze Theorem: Suppose $f(x) \le g(x) \le h(x)$, and $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x) = L$.

Then $\lim\limits_{x\to a} g(x) = L$. $\quad (*)$

e.g 7 (Hint for WW 9). Suppose $3-x \le g(x) \le 3\cos(2x)$. Compute $\lim\limits_{x\to 0} g(x)$.

Solution: Apply Squeeze Theorem with $f(x)=3-x$, $h(x)=3\cos(2x)$, $a=0$.

Actually, $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} 3-x = 3 \quad (=L)$

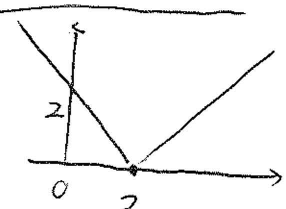
$$\lim_{x\to 0} h(x) = \lim_{x\to 0} 3\cdot\cos(2x) \xlongequal{\text{Plug in}} 3\cdot\cos 0 = 3. \quad \boxed{\text{Hint: } \cos 0 = 1}$$

Therefore, $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} h(x) = 3$, which implies

$\lim\limits_{x\to 0} g(x) = 3$ due to $(*)$.

Hint for WW 10: $|x| = \begin{cases} x & x \ge 0 \\ -x & x < 0 \end{cases}$ and $|x-2| = \begin{cases} x-2 & x \ge 2 \\ -(x-2) & x < 2 \end{cases}$

use the above piecewise expression to find the one-sided limit of the abstract-value-function.



graph of $|x-2|$.

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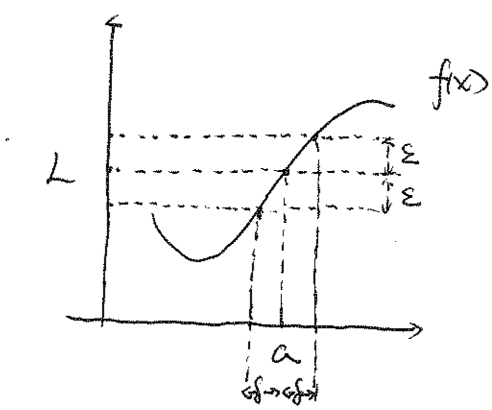
§1.7 Limit Definition.

Key points: • Error-control problem in ε-δ definition of Limit.

(ε-δ) Definition of Limit: $\lim\limits_{x\to a} f(x) = L$ means that for any output error $\varepsilon > 0$, there is an input accuracy $\delta > 0$ such that $0 < |x-a| < \delta$ forces $|f(x) - L| < \varepsilon$.

Remark: ε: Epsilon ; δ: delta are Greek letters, which are usually used for small numbers.



This gives us a precise characterization of the "intuition" approaching process about Limit in §1.5.

i.e. For given error ε, we want to FIND δ (how close x should be to a) in order that $f(x)$ is also close enough to L.

Goal: Determine δ based on the given ε.

- Linear function $f(x)$.

e.g.1. (numerically). For $f(x)=2x+1$, $L=-3$, $a=-2$, $\varepsilon=0.05$, find the largest value of δ in the ε-δ definition of a limit which ensures that $|f(x)-L|<\varepsilon$.

Solution: Plug in f, L, ε and rewrite the inequality in $|x-a|<\delta$ form.

$$|(2x+1)-(-3)|<0.05 \iff |2x+4|<0.05 \iff |x+2|<\frac{0.05}{2}=0.025.$$

In other words, if $|x-(-2)|=|x+2|<\boxed{0.025}$, then $|(2x+1)-(-3)|<0.05$.

Therefore, $\boxed{\delta=0.025}$

e.g.2. (S16) Let $f(x)=1-2x$ and $\varepsilon>0$. What is the largest value of δ such that $|x-2|<\delta$ implies $|f(x)+3|<\varepsilon$.

Solution: $|f(x)+3|=|1-2x+3|<\varepsilon \iff |4-2x|<\varepsilon$.

(Notice that $|g(x)|=|-g(x)|$ for any $g(x)$.) $\iff |2x-4|<\varepsilon$

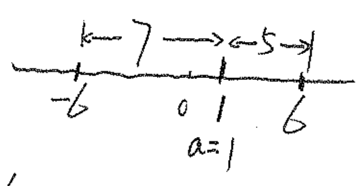
Therefore, $\delta=\frac{\varepsilon}{2}$. $\iff \boxed{|x-2|<\frac{\varepsilon}{2}}$.

Hints for WW.

e.g.3. (ww2) Find δ as above for $f(x)=\sqrt{10-x}$, $L=3$, $a=1$, $\varepsilon=1$.

Solution: Solve for x in $|\sqrt{10-x}-3|<1 \iff -1<\sqrt{10-x}-3<1 \iff 2<\sqrt{10-x}<4$

$\iff 4<10-x<16 \iff -6<-x<6 \iff -6<x<6$



Then pick the smaller distance from the endpoints to $a=1$

i.e. $\boxed{\delta=5}$ since we need $|x-1|<5$ to ensure that $-6<x<6$

ww6: Complete the square with the formula: $x^2-2a\cdot x+a^2=(x-a)^2$

Notice that $|(x-a)^2|<\varepsilon$ implies that $|x-a|<\sqrt{\varepsilon}$.